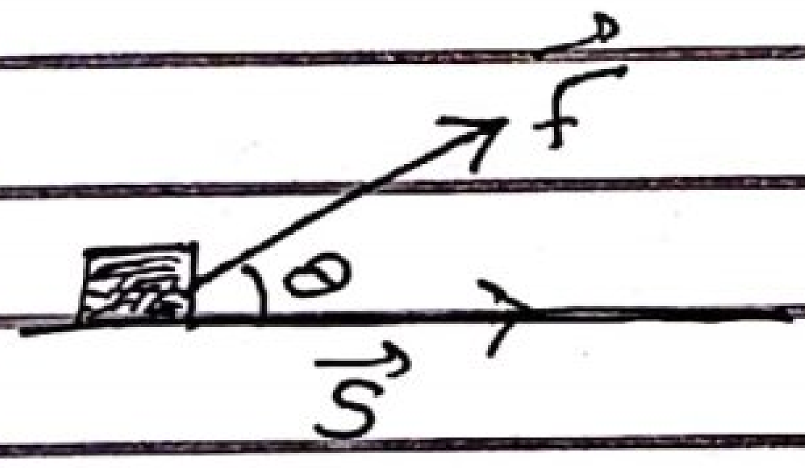
Ch. 6 Work Energy and Power

Work done by a constant Force

the work done by a constant force $\vec{F}$ is given by

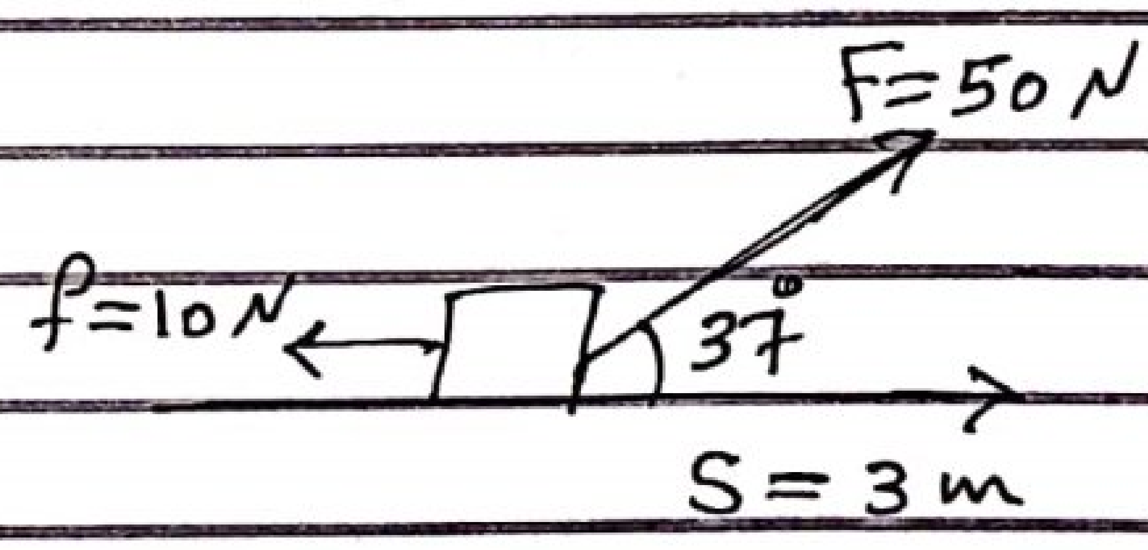
$$W = \vec{F} \cdot \vec{S} = F S \cos\theta$$



* the unit of Work is Joul (J)
  where $1 J = 1 N.m$

* the work done by frictional force is
  $W_f = -fS \quad , \theta = 180°$

Example Dragging a box

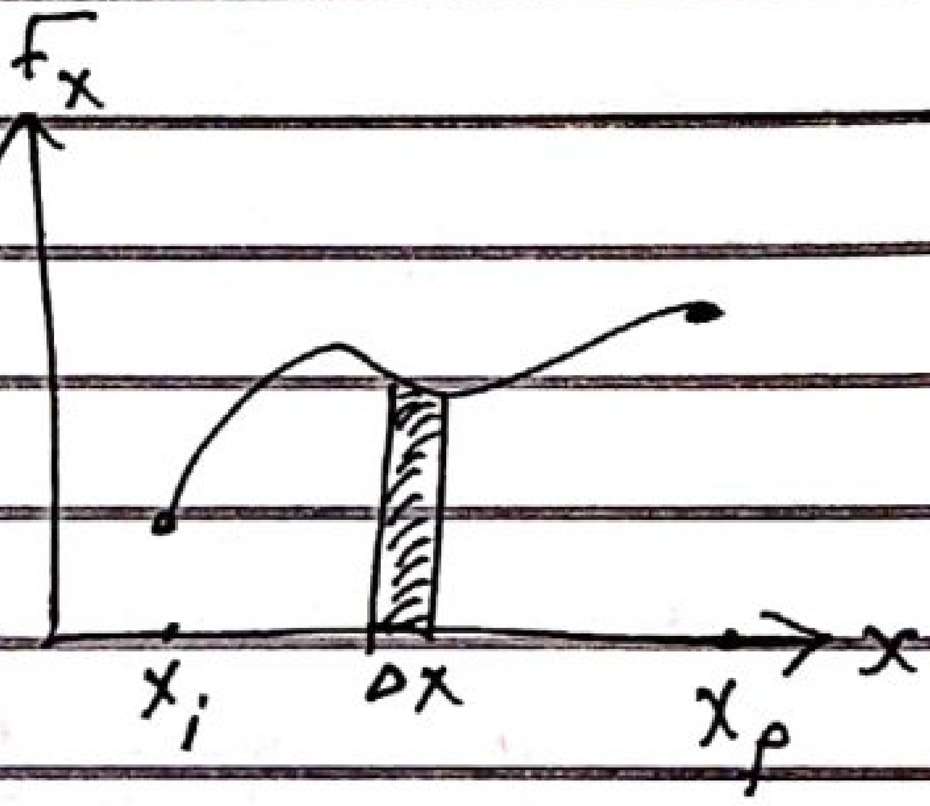


a) calculate the work done by the force F
b) calculate the work done by the frictional force f
c) determine the net work done on the box by all forces

Solution
a) $W_F = FS\cos\theta = (50)(3)(\cos 37) = 120 \, J$
b) $W_f = fS\cos 180 = -fS = -(10)(3) = -30 \, J$
c) $W_{net} = W_F + W_f = 120 - 30 = 90 \, J$

Work done by a varying Force (1 D case)

The work done on an infinitesimal displacement Δx is



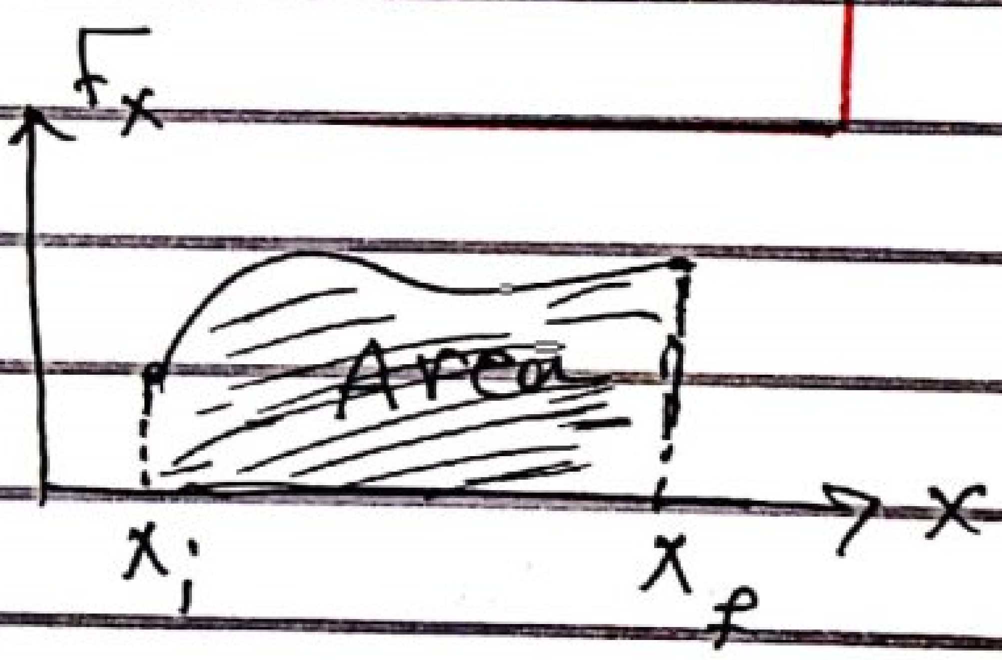
$$\Delta W = F_x \Delta x = \Delta A \quad \text{(area)}$$

the total work

$$W \sim \sum \Delta W = \sum_{x_i}^{x_f} F_x \Delta x$$

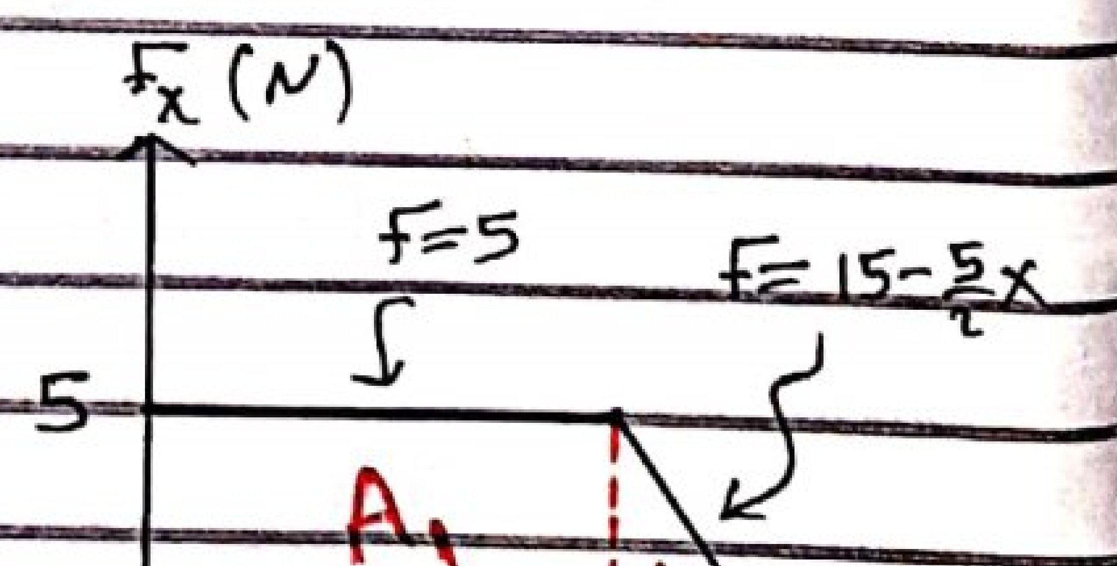
$$W = \lim_{\Delta x \to 0} \Delta W = \lim_{\Delta x \to 0} \sum_{x_i}^{x_f} F_x \Delta x = \int_{x_i}^{x_f} F_x \, dx$$

$$\boxed{W = \int_{x_i}^{x_f} F_x \, dx = \text{Area under the curve}}$$



Example

Calculate the work done by the Force F_x as the object moves from $x_i = 0$ to $x_f = 6$ m



solution

$$W = W_1 + W_2 = A_1 + A_2$$
$$= (5)(4) + \left(\tfrac{1}{2}\right)(2)(5) = 25 \text{ J}$$

or

$$W = \int_0^6 F_x \, dx = \int_0^4 5 \, dx + \int_4^6 \left(15 - \tfrac{5}{2}x\right) dx = 20 + 5 = 25 \text{ J}$$

Work and Kinetic Energy

1-D motion

$$W = \int_{x_i}^{x_f} F_x \, dx$$

but $F_x = m a_x = m \frac{dv_x}{dt} = m \frac{dv_x}{dx} \frac{dx}{dt}$ (chain rule)

$$F_x = m \frac{dv_x}{dx} v_x$$

or $F_x = m v \frac{dv}{dx}$

$$W = \int \left(m v \frac{dv}{dx}\right) dx = \int_{v_i}^{v_f} m v \, dv$$

$$W = \frac{1}{2} m (v_f^2 - v_i^2)$$

We define the kinetic energy K as

$$K = \frac{1}{2} m v^2$$

$$\Rightarrow \boxed{W = K_f - K_i = \Delta K} \text{ (Work-energy theorem)}$$

3-D motion

$$W = \int \vec{F} \cdot d\vec{s} = \int F_x \, dx + \int F_y \, dy + \int F_z \, dz$$

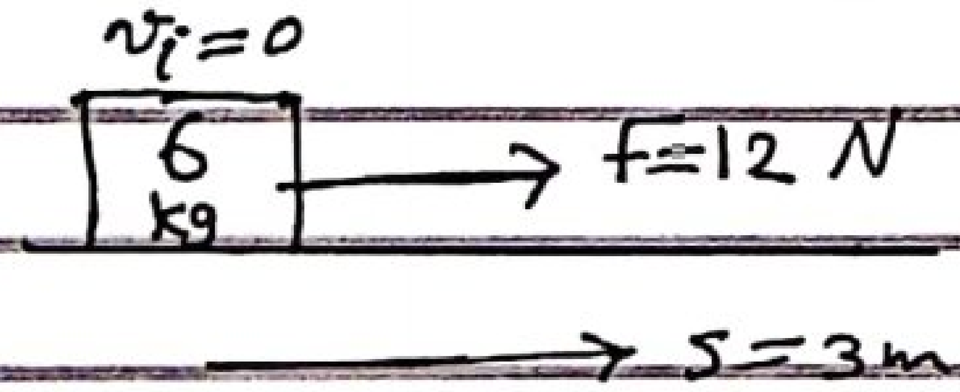
$$W = \Delta K = \frac{1}{2} m (v_f^2 - v_i^2)$$

where

$$\vec{v} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k}$$

Example

Find the speed of the block after it moves a distance 3 m on a smooth surface



$$W = FS = (12)(3) = 36\ J$$

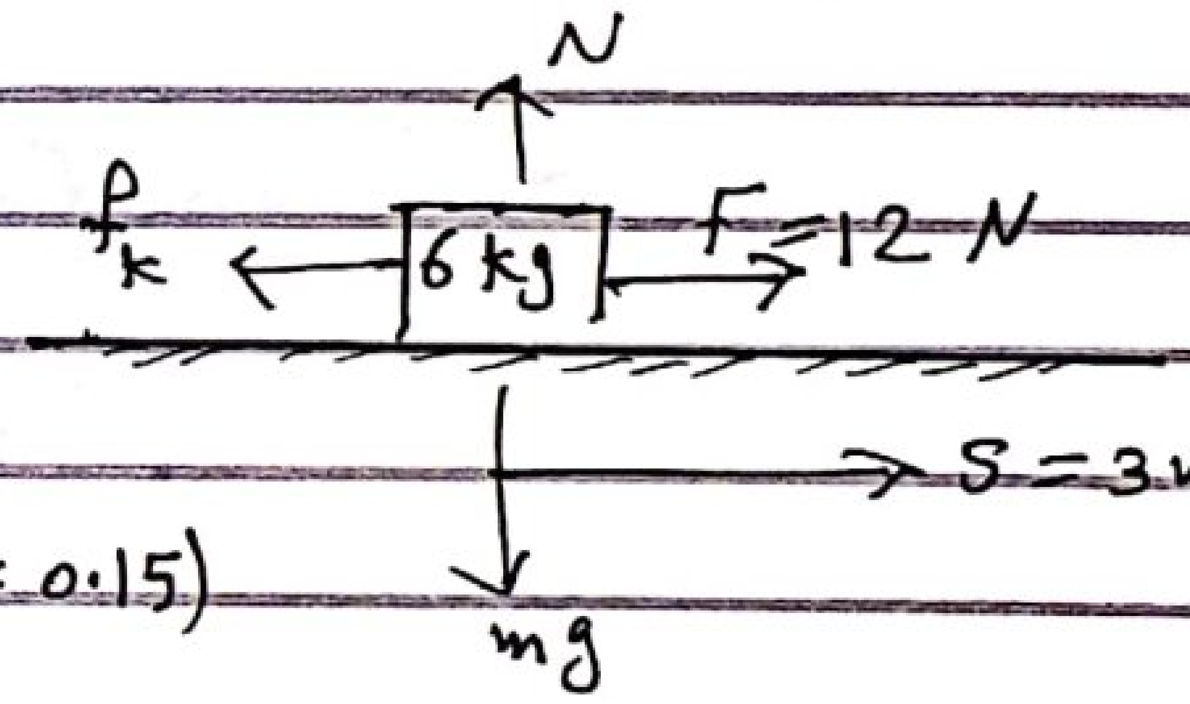
$$W = \Delta K = \frac{1}{2} m (v_f^2 - v_i^2)$$

$$36 = \frac{1}{2}(6)[v_f^2 - 0]$$

$$v_f = 3.46\ m/s$$

Example

in the previous example, what is the speed of the block after it moves a distance 3 m on a rough surface ($\mu_k = 0.15$)



$$W_F = FS = (12)(3) = 36\ J$$

$$W_f = -f_k S = -\mu_k N S = -\mu_k m g S$$

$$= -(0.15)(6)(9.8)(3) = -26.5\ J$$

$$W_{net} = W_F + W_{f_k} = 36 - 26.5 = 9.5\ J$$

$$W_{net} = \Delta K = \frac{1}{2} m (v_f^2 - v_i^2)$$

$$9.5 = \frac{1}{2}(6)[v_f^2 - 0]$$

$$v_f = 1.78\ m/s$$

Example

As an object moves a distance d upward an inclined rough plane,

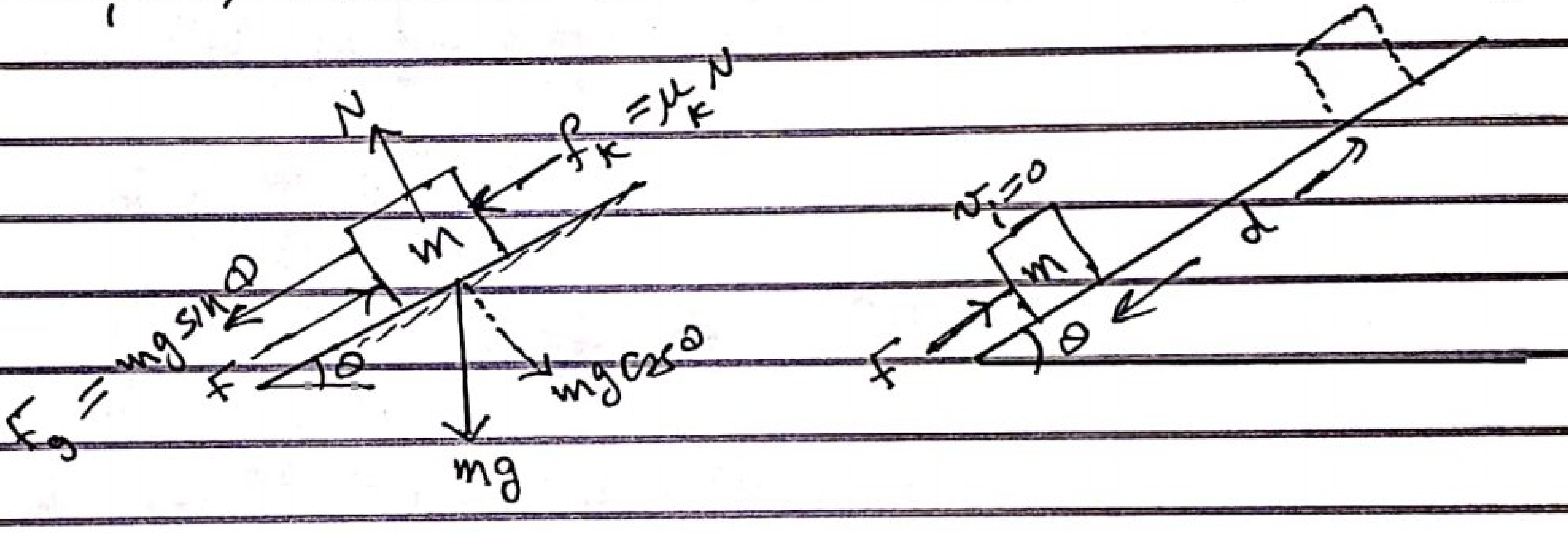
a) calculate the work done by the applied force F

b) " " " " " " force of gravity F_g

c) " " " " " " frictional force f_k

d) find the net work

e) if $v_i = 0$, what is the final speed of the object



a) $W_F = F d \cos(0) = F d$

b) $W_g = (mg\sin\theta)(d)\cos 180 = -mgd\sin\theta$

c) $W_f = f_k d \cos 180 = -f_k d = -\mu_k mg\cos\theta\, d$

d) $W_{net} = Fd - mgd\sin\theta - \mu_k mgd\cos\theta$

e) $W_{net} = \Delta K = \frac{1}{2} m (v_f^2 - v_i^2) = \frac{1}{2} m v_f^2$

$$v_f = \sqrt{\frac{2}{m} W_{net}}$$

$$= \sqrt{\frac{2d}{m}[F - mg\sin\theta - \mu_k mg\cos\theta]}$$

* if $F = 15\,N$, $d = 1\,m$, $\theta = 25°$, $m = 1.5\,kg$, $\mu_k = 0.3$

then $W_F = 15J$, $W_g = -6.2J$, $W_f = -4J$, $W_{net} = 4.8J$

$v_f = 2.5\ m/s$

Example

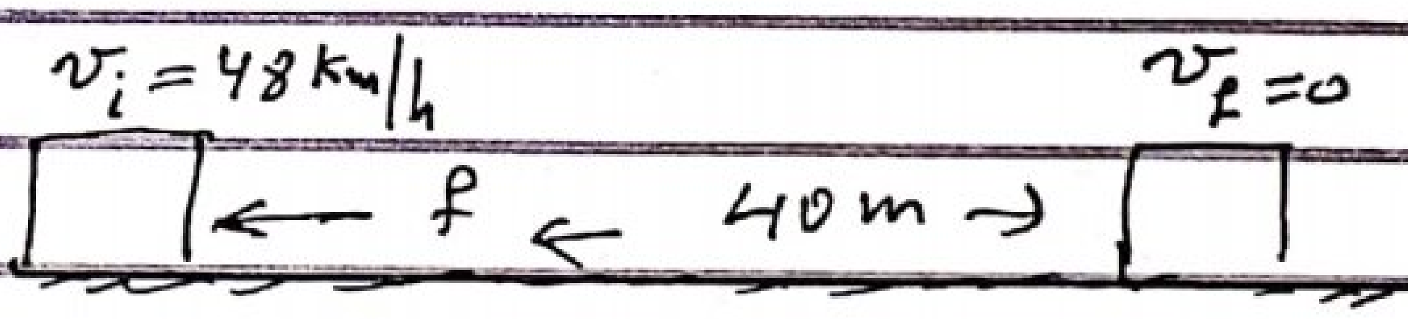
An object given an initial velocity of 48 Km/hr and passes a distance of 40 m before coming to rest due to the friction, if the initial velocity changes to 96 Km/hr, what is the distance travelled to come to rest?

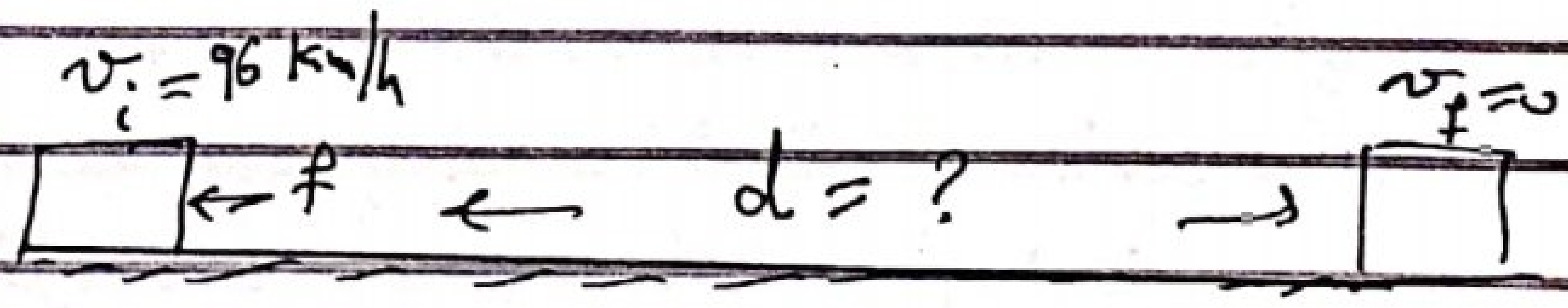
$$W_f = -fd = \frac{1}{2}m(v_f^2 - v_i^2) = -\frac{1}{2}m v_i^2$$

$$d = \frac{m}{2f} v_i^2$$

$$\frac{d_1}{d_2} = \frac{v_{i1}^2}{v_{i2}^2} \Rightarrow d_2 = \frac{v_{i2}^2}{v_{i1}^2} d_1$$

$$d_2 = \left(\frac{96}{48}\right)^2 (40) = 4(40) = 160$$





Power

- the average power is defined as the ratio of the work done to the time interval

$$\bar{P} = \frac{\Delta W}{\Delta t}$$

- the instantaneous power

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}$$

$$P = \frac{dW}{dt} = \frac{d}{dt}(\vec{F} \cdot \vec{S}) = \vec{F} \cdot \frac{d\vec{S}}{dt} = \vec{F} \cdot \vec{v} \quad \text{for constant } \vec{F}$$

the unit of power is Watt (W), where
$1\ W = J/s$

Example
A 2 kg object moves in a straight line with an initial speed of 4 m/s, It accelerates uniformly to a final speed of 7 m/s in 15 s. Calculate the average power delivered to the object

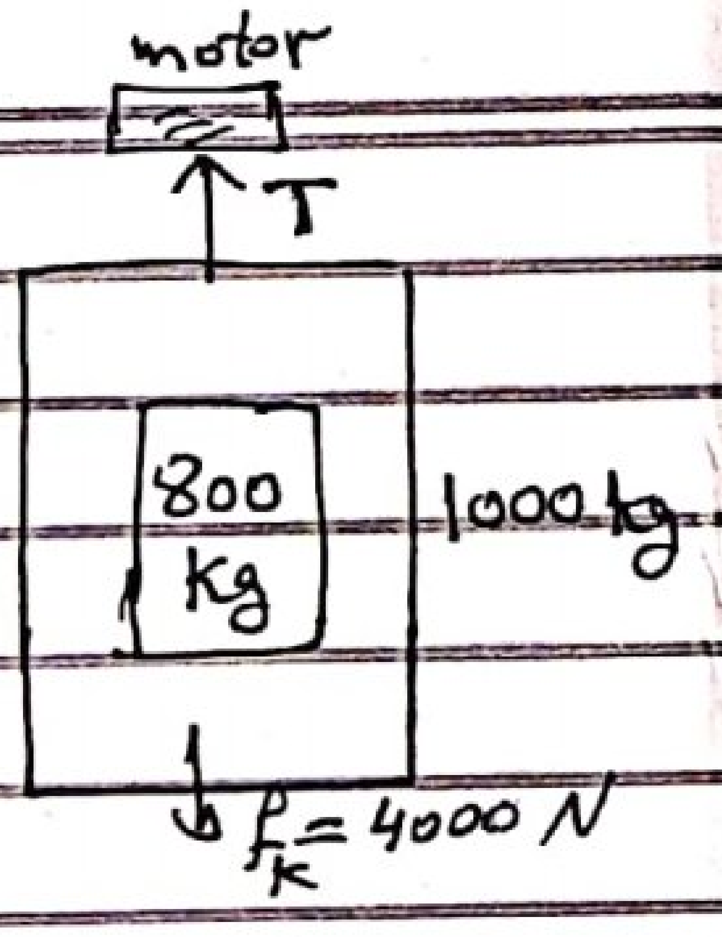
$$\bar{P} = \frac{\Delta W}{\Delta t} = \frac{\Delta K}{\Delta t} = \frac{\frac{1}{2} m (v_f^2 - v_i^2)}{\Delta t}$$

$$\bar{P} = \frac{\frac{1}{2}(2)\left[(7)^2 - (4)^2\right]}{15} = 2.2\ W$$

Example
An object moves along the x-axis with an instantaneous speed of 5 m/s under the influence of a force $\vec{F} = (3\hat{i} + 4\hat{j})\,N$. What is the instantaneous power delivered by $\vec{F}$.

$$P = \vec{F} \cdot \vec{v} = (3\hat{i} + 4\hat{j}) \cdot (5\hat{i}) = 15\ W$$

Example motion of an elevator



a) What must be the minimum power delivered by the motor to lift the elevator at a constant speed of 3 m/s

b) What power must the motor deliver at any instant if it is designed to provide an upward acceleration of 1 m/s²

solution

a) $v = 3\ m/s = \text{constant} \Rightarrow a = 0$

$$\sum F = ma = 0$$

$$T - mg - f = 0$$

$$T = mg + f = (1800)(9.8) + 4000 = 2.16 \times 10^4\ N$$

$$P = \vec{T} \cdot \vec{v} = (2.16 \times 10^4)(3) = 6.49 \times 10^4\ W$$

b) $\sum F = ma$, $a = \text{const.}$

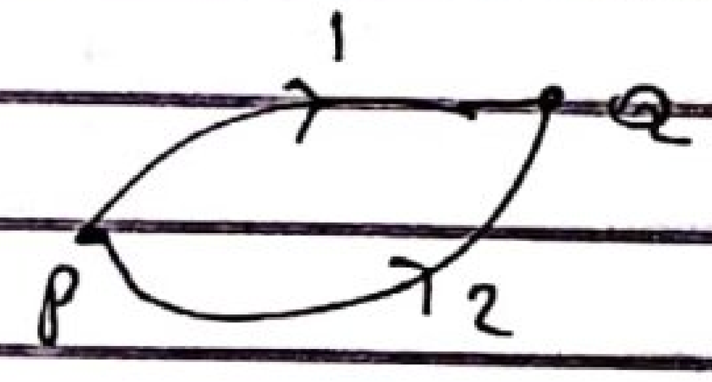
$$T - mg - f = ma$$

$$T = mg + f + ma$$
$$= (1800)(9.8) + 4000 + (1800)(1) = 2.34 \times 10^4\ N$$

$$P = \vec{T} \cdot \vec{v} = 3.34 \times 10^4\ v$$

Conservative Force

a Force is conservative if the work done by that force on a particle between two points is independent of the path the particle takes between the points



$$(W_{PQ})_1 = (W_{PQ})_2$$

$$(W_{PQ})_1 = -(W_{QP})_2$$

$$(W_{PQ})_1 + (W_{QP})_2 = 0$$

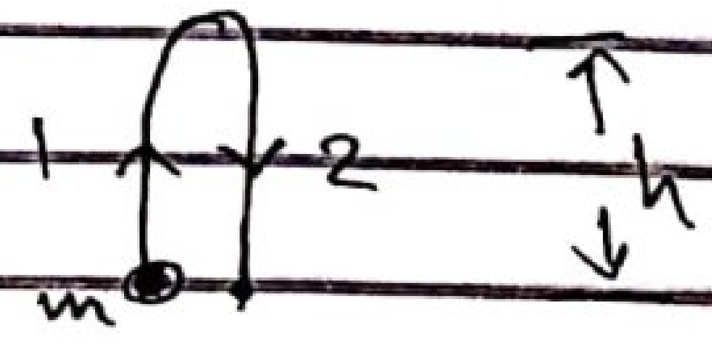
* this means that the total work done by a conservative force on a particle is zero when the particle moves around a closed path and returns to its initial position

$$W_{tot} = \oint \vec{F} \cdot d\vec{s} = 0$$

Examples of conservative forces

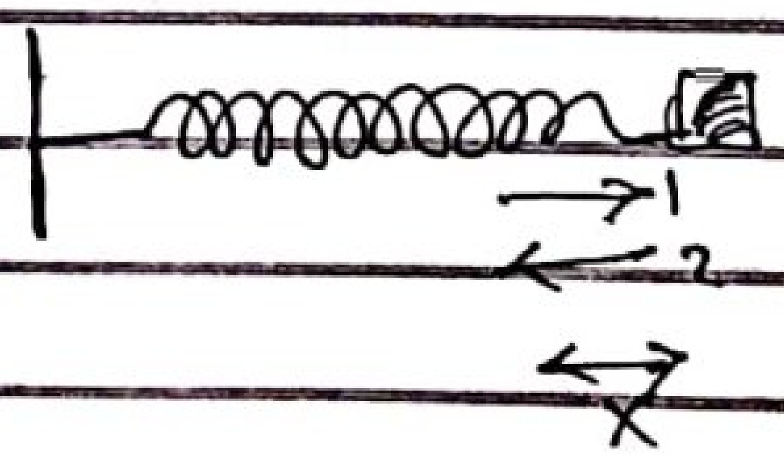
① the Force of gravity

$$W_1 + W_2 = -mgh + mgh = 0$$



② spring Force

$$W_1 + W_2 = 0$$



where

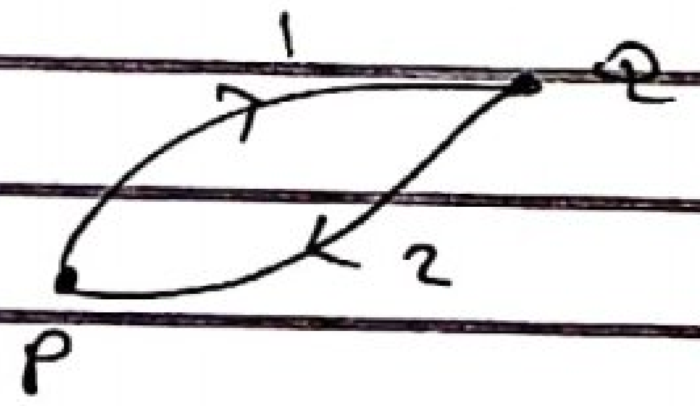
$$W_1 = \frac{1}{2}k(x_i^2 - x_f^2)$$

$$W_2 = \frac{1}{2}k(x_f^2 - x_i^2)$$

Non Conservative Force

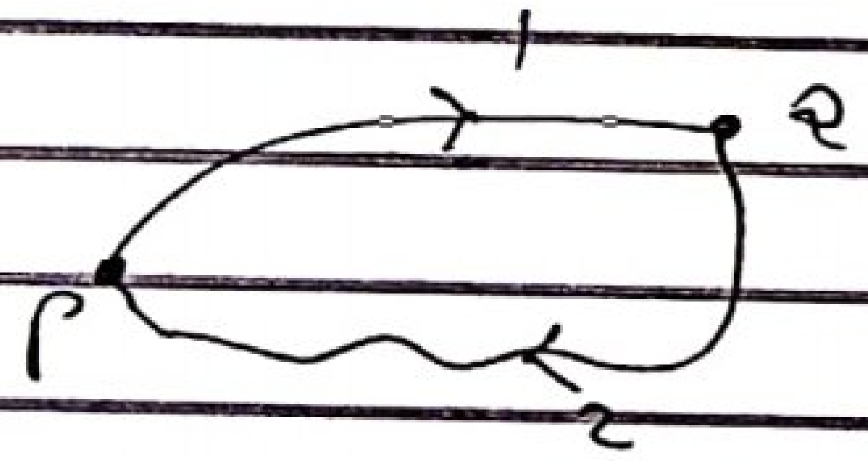
A force is nonconservative if the work done by the force on a particle moving between two points depends on the path taken

$$(W_{PQ})_1 \neq (W_{PQ})_2$$



An Example of nonconservative force is the frictional force

$$W_1 = -f d_1 \quad \Rightarrow W_1 + W_2 \neq 0$$
$$W_2 = -f d_2$$



here f is contant but d is not constant

Potential Energy

We define the potential energy such that the work done by the conservative force equals the decrease in potential energy

$$W_c = \int_{x_i}^{x_f} F_x \, dx = -\Delta U$$

or $$U_f - U_i = -\int_{x_i}^{x_f} F_x \, dx$$

Conservation of Mechanical Energy

From the work-energy theorem $W_c = \Delta K$
and from the definition of potential energy $W_c = -\Delta U$

That is $W_c = \Delta K = -\Delta U$

$$\boxed{\Delta K + \Delta U = 0}$$

or

$$K_f - K_i + U_f - U_i = 0$$

$$U_i + K_i = U_f + K_f$$

$$\boxed{E_i = E_f}$$ conservation law of mech. energy

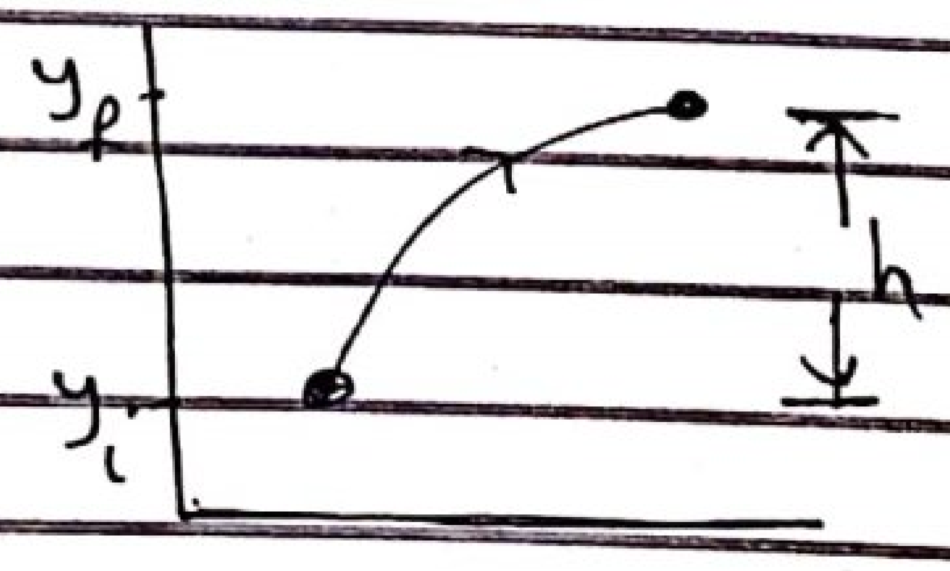
where E is the total mechanical energy

$$E = U + K$$

Gravitational potential energy near the earth's surface

$$U_f - U_i = -\int_{y_i}^{y_f} F_y\, dy = -\int -mg\,dy$$

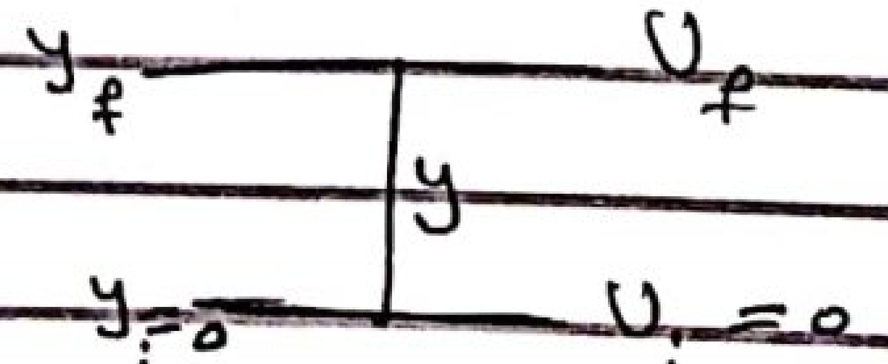
$$U_f - U_i = mg(y_f - y_i)$$



choose $U_i = 0$ at $y_i = 0$, then

$$U_f = mgy_f$$

or $$\boxed{U_g = mgy}$$

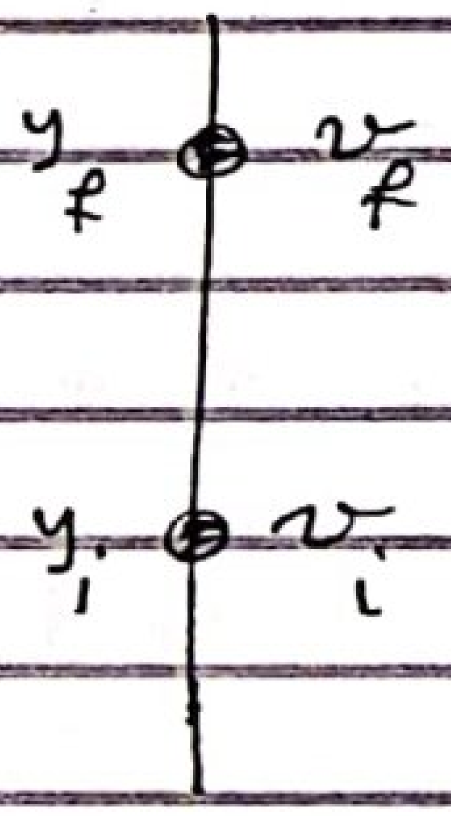


Conservation of mechanical energy of a freely falling body

$$E_i = E_f$$

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2} m v_i^2 + m g y_i = \frac{1}{2} m v_f^2 + m g y_f$$



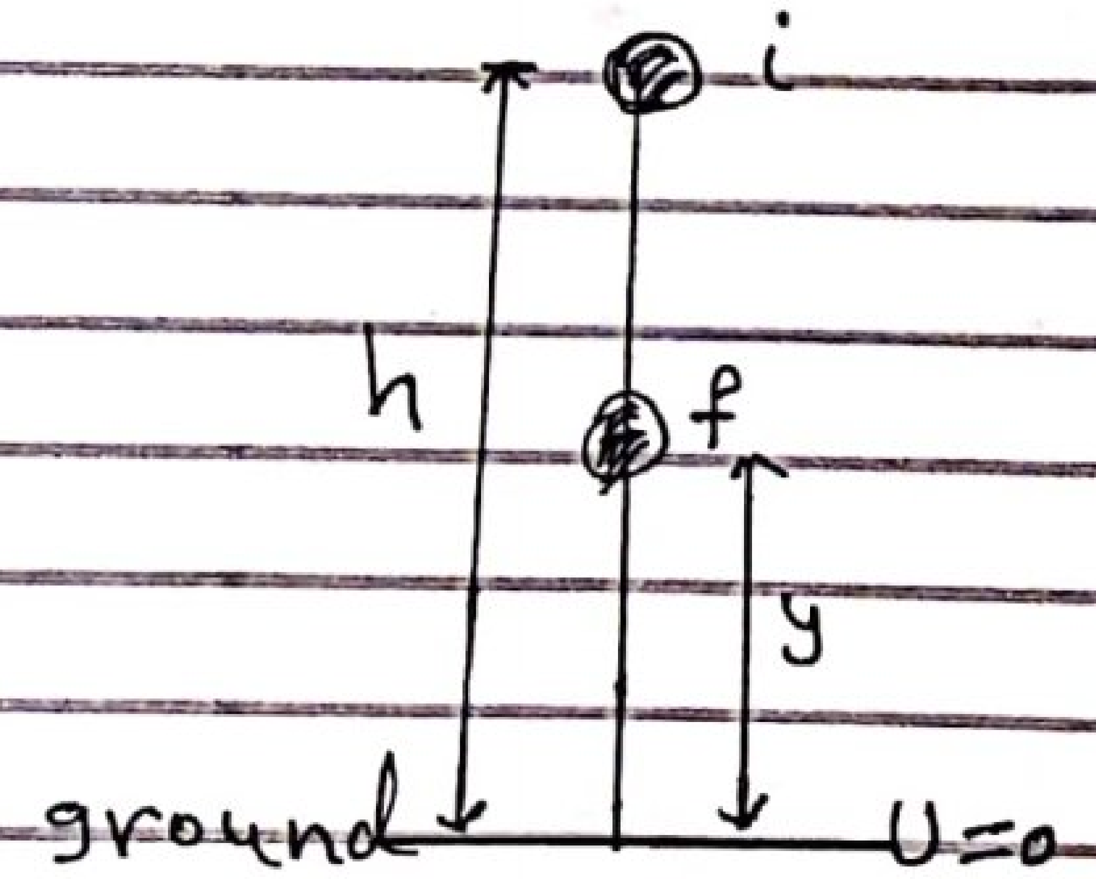
Example

A ball of mass m is dropped from a height h above the ground. Determine

a) the speed of the ball when it is a height y above the ground

b) the speed of the ball at y if it is given an initial speed v_i at the initial altitude h

solution



a) $E_i = E_f$

$$K_i + U_i = K_f + U_f$$

$$0 + mgh = \frac{1}{2} m v_f^2 + mgy$$

$$v_f = \sqrt{2g(h-y)}$$

b) $E_i = E_f$

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2} m v_i^2 + mgh = \frac{1}{2} m v_f^2 + mgy$$

$$v_f = \sqrt{v_i^2 + 2g(h-y)}$$

Example (the Pendulum)

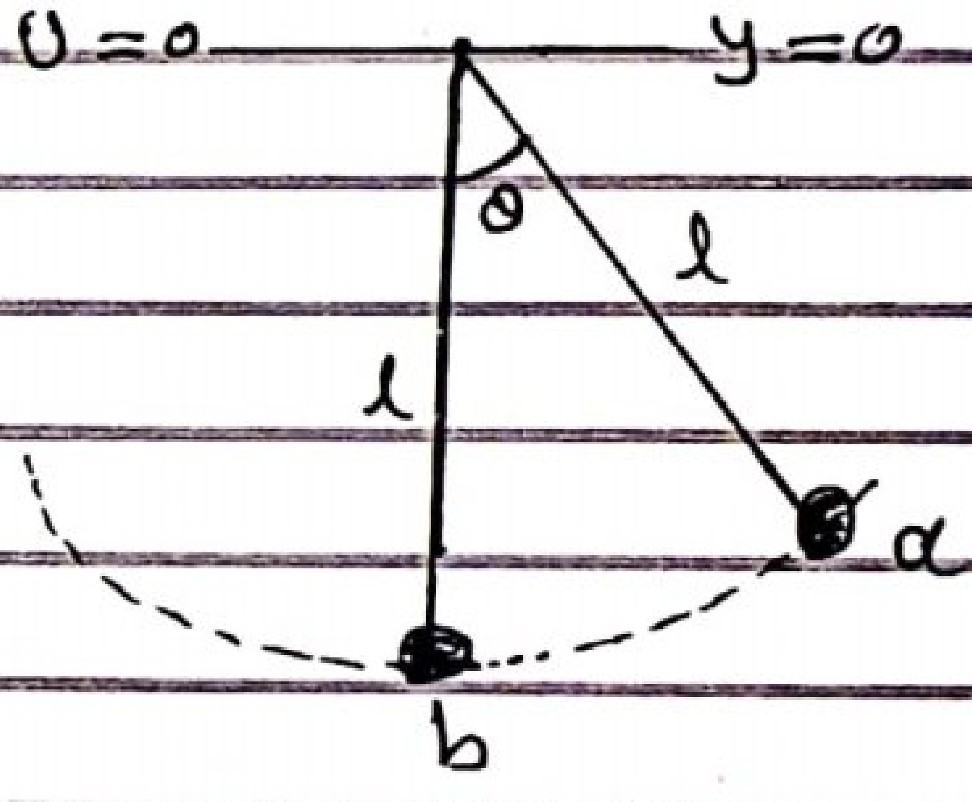
A pendulum consists of a sphere of mass m attached to a light cord of length l. The sphere is released from rest when the cord makes an angle θ with the vertical

a) find the speed of the sphere when it is at the lowest point (b)

b) what is the tension T in the cord at b

c) find the speed and tension at b when $l = 2\,m$, $m = 0.5\,kg$, $\theta = 30°$

solution

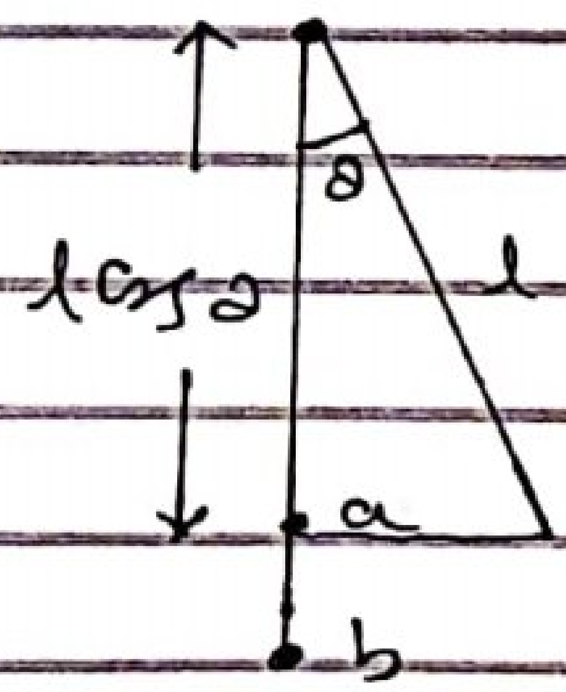


a) $E_a = E_b$

$$K_a + U_a = K_b + U_b$$

$$0 - mgl\cos\theta = \frac{1}{2}mv_b^2 - mgl$$

$$v_b = \sqrt{2gl(1-\cos\theta)}$$



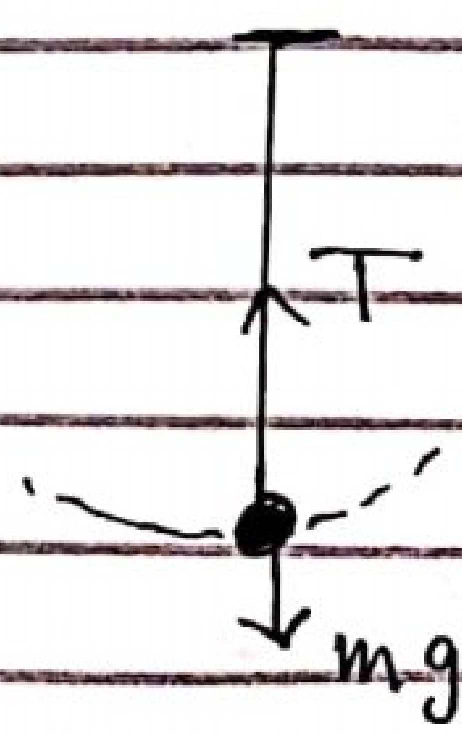
b) $\Sigma F_r = m a_c$

$$T_b - mg = m\frac{v_b^2}{l}$$

$$T_b = mg + m\frac{v_b^2}{l}$$

$$= mg + \frac{m}{l}\,2gl(1-\cos\theta)$$

$$T_b = mg(3 - 2\cos\theta)$$

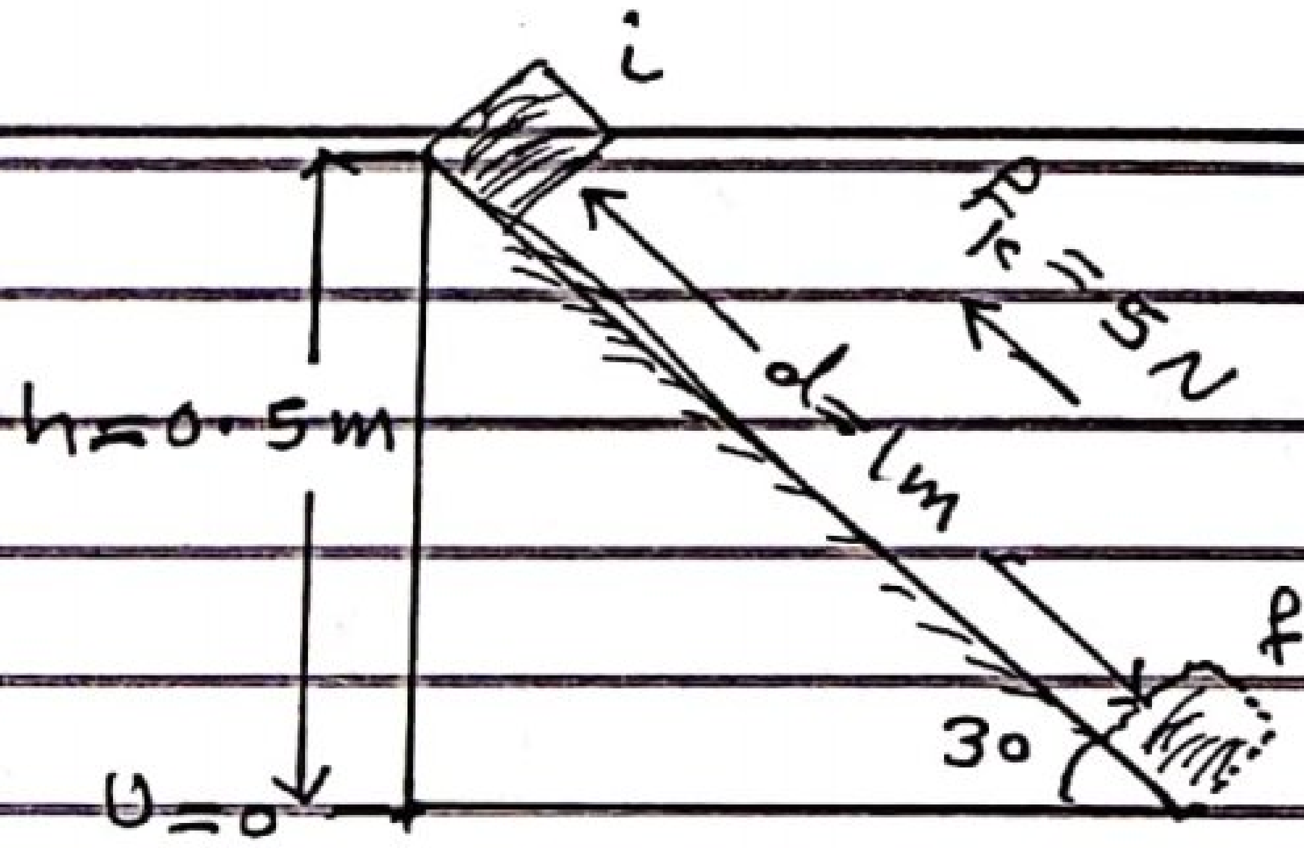


c) $v_b = \sqrt{2(9.8)(2)\left[1 - \frac{\sqrt{3}}{2}\right]} = 2.3\,m/s$

$$T_b = 0.5(9.8)\left[3 - 2\frac{\sqrt{3}}{2}\right] = 6.2\,N$$

solution

a) $E_i = U_i + K_i = mgh + 0$



$= 3(9.8)(0.5) = 14.7\,J$

$E_f = U_f + K_f = 0 + \frac{1}{2} m v_f^2$

$= \frac{3}{2} v_f^2$

$W_{nc} = -f_k d = -(5)(1) = -5\,J$

$W_{nc} = E_f - E_i$

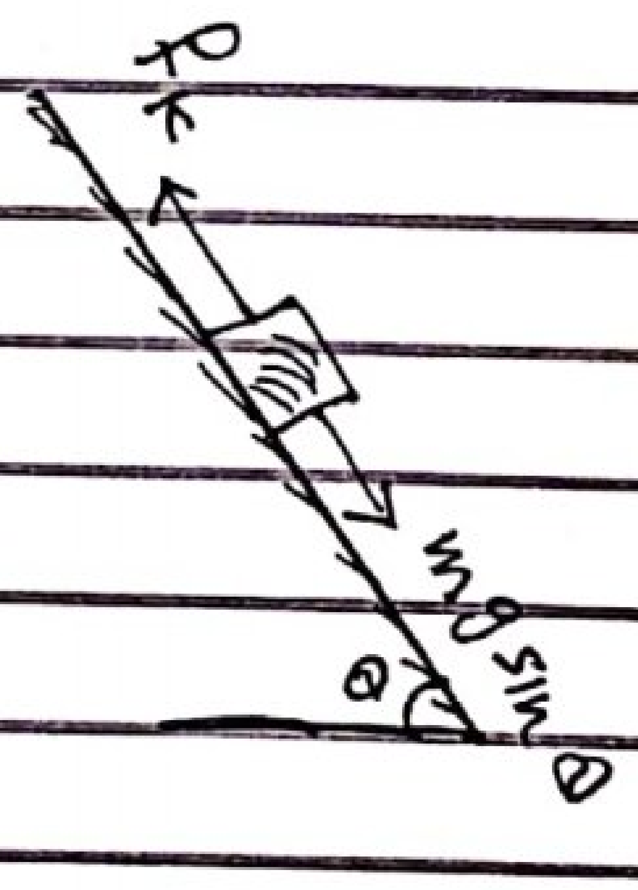
$-5 = \frac{3}{2} v_f^2 - 14.7 \Rightarrow v_f = 2.54\ m/s$

b) Newton's 2nd law : $\Sigma F = ma$

$mg\sin\theta - f_k = ma$

$(3)(9.8)(\sin 30) - 5 = 3a$

$\Rightarrow a = 3.23\ m/s^2$ = constant



$v_f^2 = v_i^2 + 2ad$

$= 0 + 2(3.23)(1)$

$v_f = 2.54\ m/s$

c) for frictionless surface $W_{nc} = 0$

$E_i = E_f$

$K_i + U_i = K_f + U_f$

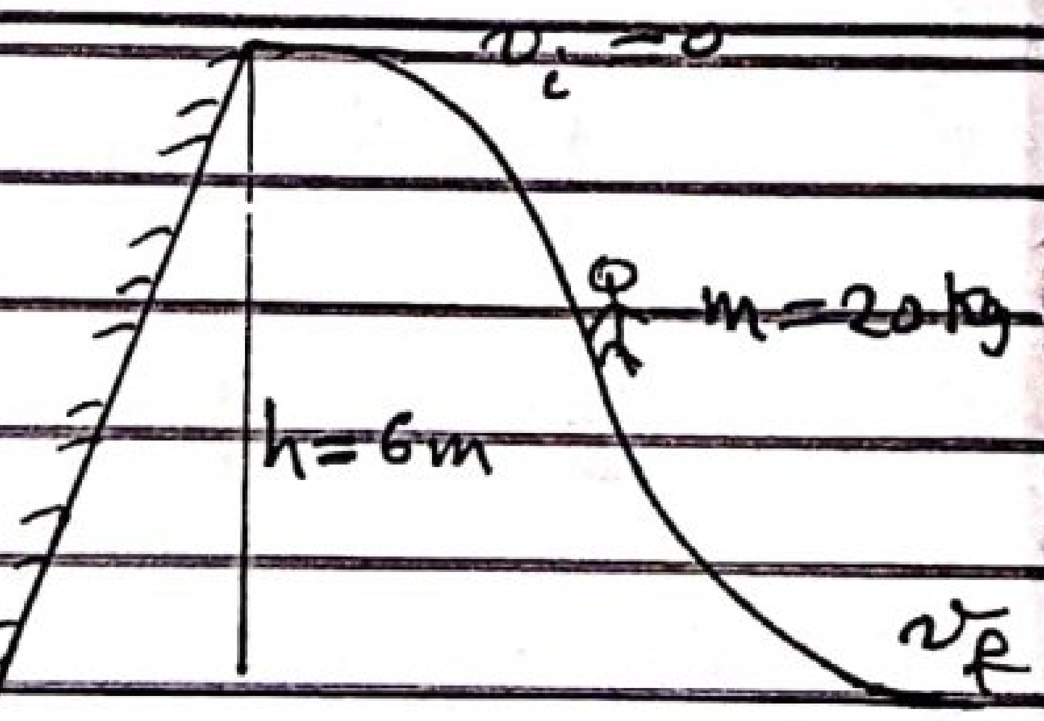
$0 + mgh = \frac{1}{2} m v_f^2 + 0 \Rightarrow v_f = \sqrt{2gh}$

~~$(3)(9.8)(0.5) = \frac{1}{2}(3) v_f^2$~~

$= \sqrt{2(9.8)(0.5)}$

$= 3.13\ m/s$

Example



a) determine the speed of the child at the bottom

b) if there were a frictional force, what would be the work done by this force if he reaches the bottom at a speed of 8 m/s

$U = 0$

Solution

a) $E_i = E_f$

$K_i + U_i = K_f + U_f$

$0 + mgh = \frac{1}{2} m v_f^2 + 0$

$v_f = \sqrt{2gh} = \sqrt{2(9.8)(6)} = 10.8 \text{ m/s}$

b) $W_{nc} = E_f - E_i$

$E_f = \frac{1}{2} m v_f^2$

$E_i = mgh$

$W_{nc} = \frac{1}{2} m v_f^2 - mgh$

$= \frac{1}{2}(20)(8)^2 - (20)(9.8)(6)$

$W_{nc} = -536 \text{ J}$